Multiagent decision-making and control

Potential games

Maryam Kamgarpour

Professor of Engineering (IGM, STI), EPFL

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Course topics

1. Static games
2. Zero-sum games
3. Potential games
4. Dynamic games, dynamic programming principle
5. Dynamic games, dynamic programming for games
6. Dynamic games, linear quadratic games, Markov games
7. Convex games, Nash equilibria characterization
8. Convex games, Nash equilibria computation
9. Auctions
10. Bayesian games
11. Learning in games
12. Extensive form games
13. Feedback games in extensive form
14. Final project presentations

N-player non-zero-sum games

- N players
- Player i can choose one among m_i pure actions.

$$\Gamma_i = \{\gamma_i^{(1)}, \gamma_i^{(2)}, \ldots, \gamma_i^{(m_i)}\}$$

- Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_N$. The cost (utility/payoff if maximizers) for Player i is given by $J_i : \Gamma \to \mathbb{R}$:

$$J_i(\gamma_1, \gamma_2, \ldots, \gamma_N) = J_i(\gamma_i, \gamma_{-i}),\ \gamma_{-i} = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_N)$$

Pure Nash equilibrium

A strategy profile $\gamma^* = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma$ is a pure Nash equilibrium if for every player i

$$J_i(\gamma_i^*, \gamma_{-i}^*) \leq J_i(\gamma_i', \gamma_{-i}^*), \qquad \gamma_i' \in \Gamma_i$$

- Multiple Nash equilibria are possible (non interchangeable)
- Randomized play $\to$ mixed strategies, mixed Nash equilibria

Best-response

- Let γ_{-i} be the pure strategy profile of all players other than i
- Let $\Gamma_{-i} = \Gamma_1 \times \cdots \times \Gamma_{i-1} \times \Gamma_{i+1} \times \Gamma_N$ be the pure strategy spaces of players other than i.
- How should player i choose her strategy?

Best-response map

The best-response of player i is the set $R_i(\gamma_{-i}) \subseteq \Gamma_i$ such that

$$\gamma_i \in R_i(\gamma_{-i}) \iff J_i(\gamma_i, \gamma_{-i}) \leq J_i(\gamma_i', \gamma_{-i}) \qquad \forall \gamma_i' \in \Gamma_i.$$

In other words: $R_i(\gamma_{-i}) := \operatorname{argmin}_{\gamma_i \in \Gamma_i} J_i(\gamma_i, \gamma_{-i})$.

Recall what we had shown before regarding $R_i : \Gamma_{-i} \to 2^{\Gamma_i}$:

- $R_i(\gamma_{-i})$ is a set, not necessarily a singleton (one element only)
- $R_i(\gamma_{-i})$ is never empty (why?)

Review: best-response and Nash equilibrium

Proposition

A strategy profile $\gamma^* = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma$ is a pure Nash equilibrium if and only if $\gamma_i^* \in R_i(\gamma_{-i}^*)$ for every player i.

Recall the fixed-point characterization of a Nash equilibrium:

- Consider a set valued map R such that when $\gamma \in \Gamma$, $R(\gamma) := [R_1(\gamma_{-1}), R_2(\gamma_{-2}), \ldots, R_N(\gamma_{-N})] \subset \Gamma$.
- Prove the above (we did it in Lecture 1).
- Remark: we used this characterization of Nash equilibria in proof of existence of mixed strategy Nash equilibria.
- But what can we do to find a Nash equilibrium?

Best-response dynamics

Iterative best-response update

Consider an initial pure strategy profile $\gamma(0) = (\gamma_1(0), \gamma_2(0), \dots, \gamma_n(0))$.
Step $k = 0, \dots$:

1. If $\gamma(k)$ is a pure Nash equilibrium $\rightarrow$ **stop**
2. Else there exists a player i, and $\tilde{\gamma}_i \neq \gamma_i(k)$ such that $\tilde{\gamma}_i \in R(\gamma_{-i}(k))$.
3. Update: $\gamma(k+1) := (\tilde{\gamma}_i, \gamma_{-i}(k))$.
4. $k = k + 1$, goto step 1.

- Verify that the best-response dynamic terminates if and only if $\bar{\gamma}$ is a pure strategy Nash equilibrium.
- Under which conditions on the game the above dynamics converges?

Examples of best-response dynamics behavior

In which of these games best-response dynamics converge? players are maximizers

here, it converges to an NE,

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{cc} A & B \\ \left[\begin{array}{cc} (3,2) & (1,1) \\ (0,0) & (2,3) \end{array}\right] \end{array}$$

depending on initial condition it converge to a different NE

Clearly, it does not converge if a pure Nash equilibrium **does not exist**.

doesn't converge

$$\begin{array}{c} \\ \text{heads} \\ \text{tail} \end{array} \begin{array}{cc} \text{heads} & \text{tail} \\ \left[\begin{array}{cc} (1,-1) & (-1,1) \\ (-1,1) & (1,-1) \end{array}\right] \end{array}$$

Does it converge to a pure Nash equilibrium if it exists?

converges if initialized suitably

$$A = \begin{array}{c} \\ U \\ M \\ B \end{array} \begin{array}{ccc} L & M & R \\ \left[\begin{array}{ccc} (2,2) & (-2,-2) & (-2,-2) \\ (-2,-2) & (1,-1) & (-1,1) \\ (-2,-2) & (-1,1) & (1,-1) \end{array}\right] \end{array}$$

In this lecture

Potential games: A class of N-player general-sum games for which

- a pure Nash equilibrium is guaranteed to exist
- best-response dynamics converges

Potential functions and potential games

Ordinal

A function $P : \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_N \to \mathbb{R}$ is an **ordinal potential function** if for every player i and every γ_{-i},

$$J_i(\gamma_i', \gamma_{-i}) - J_i(\gamma_i'', \gamma_{-i}) \overset{>}{\cancel{\geq}} 0 \quad \text{iff} \quad P(\gamma_i', \gamma_{-i}) - P(\gamma_i'', \gamma_{-i}) \overset{>}{\cancel{\geq}} 0$$

for every $\gamma_i', \gamma_i'' \in \Gamma_i$.

Observe: when player i chooses a best-response, the potential increases

Exact potential function

A function $P : \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_N \to \mathbb{R}$ is an **exact potential function** if for every player i and every γ_{-i},

$$J_i(\gamma_i', \gamma_{-i}) - J_i(\gamma_i'', \gamma_{-i}) = P(\gamma_i', \gamma_{-i}) - P(\gamma_i'', \gamma_{-i})$$

for every $\gamma_i', \gamma_i'' \in \Gamma_i$.

A game is an **(ordinal/exact) potential game** if it admits an (ordinal/exact) potential function.

[Monderer and Shapley (1996). *Potential Games*. Games and Economic Behavior. 14: 124–143.]

Example 1 - verifying if a function is exact potential for a game

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} (3, 2) & (1, 1) \\ (0, 0) & (2, 3) \end{bmatrix} \end{array}$$

- Verify this function is an exact potential for the above game

$$P = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Exercise 1 - verifying if a function is an ordinal potential for a game

$$\begin{array}{cc} & \begin{array}{cc} \text{L} & \text{R} \end{array} \\ \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{bmatrix} (0,2) & (0,3) \\ (1,0) & (0,1) \end{bmatrix} \end{array}$$

- Verify this function is an ordinal potential for the above game

$$P = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$$

From Voorneveld and Norde paper, Figure 2

Exercise 2 - verifying if a game is ordinal potential

Show that the following game is not an ordinal potential game.

$$\begin{array}{cc} & \begin{array}{cc} \quad L \quad & \quad R \quad \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} (1,0) & (2,0) \\ (2,0) & (0,1) \end{bmatrix} \end{array}$$

Hint: show there cannot be any function that satisfies the inequalities in the definition of an ordinal potential function.

(you can check your work by seeing Monderer and Shapley paper, just after Theorem 2.4)

Deriving a potential function

- Write a set of equations for deriving an exact potential function for *stag hunt*

$P_{11} = x_1$
$P_{12} = x_2$
$P_{21} = x_3$
$P_{22} = x_4$

$$\begin{array}{cc|cc} & & L & R \\ & & \text{Stag} & \text{Hare} \\ T & \text{Stag} & (10, 10) & (0, 4) \\ B & \text{Hare} & (4, 0) & (4, 4) \end{array}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Potential function equations

$$\left.\begin{aligned} J_1(T,L) - J_1(B,L) &= P_{11} - P_{21} = 6 \\ J_2(T,L) - J_2(T,R) &= P_{11} - P_{12} = 6 \\ J_1(T,R) - J_1(B,R) &= P_{12} - P_{22} = -4 \\ J_2(B,L) - J_2(B,R) &= P_{21} - P_{22} = -4 \end{aligned}\right\} \Rightarrow \quad Ax = b$$

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

  - Verify the best-response dynamics converges to a Nash equilibrium (to which?)

- Try the same exercise for the game *matching pennies*.

$$b = \begin{bmatrix} 6 \\ 6 \\ -4 \\ -4 \end{bmatrix}$$

$$P = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

① $Ax = b$ has a unique solution

$\Longleftrightarrow$ A is invertible $\Longleftrightarrow$ A doesn't have 0 eigenvalue.

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

associated eigenvector

$Av = 0 \cdot v$, $\quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, 0 is an eigenvalue of A

system will not have a unique solution

Does it have a solution at all?

$Ax = b$ has a solution $\iff b \in \text{Range}(A)$

$x_0 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 9 \end{bmatrix}$ is a solution (verify)

furthermore, $x_0 + c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $c \in \mathbb{R}$ is also a solution

$\Rightarrow \quad P = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$

Existence of pure Nash equilibrium

Proposition

(1) If a game admits a potential function, then it has a pure strategy Nash equilibrium. (2) Furthermore, the best response dynamics converge.

- Provides a **computation method** and an intuition for **repeated games**
- These iterations converge to a Nash equilibrium that depends on the **initial conditions**
- It does not converge only to admissible Nash equilibria (see coordination game)

Proof: see board.

Let $\gamma^* \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N$ be an optimizer (minimizer) of $P : \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N \to \mathbb{R}$

then by definition

$$\Longleftrightarrow P(\gamma^*) \leq P(\gamma), \quad \forall \gamma \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N$$

$$\Longrightarrow P(\gamma_i^*, \gamma_{-i}^*) \leq P(\gamma_i, \gamma_{-i}^*), \quad \forall \gamma_i \in \Gamma_i$$

by definition of potential game

$$\Longleftrightarrow J_i(\gamma_i^*, \gamma_{-i}^*) \leq J_i(\gamma_i, \gamma_{-i}^*), \quad \forall \gamma_i \in \Gamma_i$$

since above holds for any player, i, $\forall i$,

γ^* is a Nash equilibrium.

Another characterization of potential games

- A **path** in Γ is a sequence

$$\mathcal{P} = (\gamma(0), \gamma(1), \dots, \gamma(M)), \qquad \gamma(k) \in \Gamma$$

such that for every k, there exists a unique player i_k such that

$$\gamma(k-1) = (\gamma_{i_k}, \gamma_{-i_k}) \qquad \to \gamma(k) = (\gamma'_{i_k}, \gamma_{-i_k}), \qquad \text{with } \gamma_{i_k} \neq \gamma'_{i_k}$$

- Consider any **finite path** $\mathcal{P} = (\gamma(0), \gamma(1), \dots, \gamma(M))$.
- A path is **closed** if $\gamma(0) = \gamma(M)$.
- A path is **simple** if $\gamma(k) \neq \gamma(k')$, for every k, k' (except $\gamma(0)$ and $\gamma(M)$).

Checking whether a game is potential

Define

$$I(\mathcal{P}) := \sum_{k=1}^{M} [J_{i_k}(\gamma(k)) - J_{i_k}(\gamma(k-1))]$$

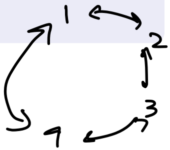
where i_k is the player that changes its strategy at step k.

Theorem 2.8 from [Monderer & Shapley]

Consider a finite game. Then the following statements are equivalent:

1. The game admits an exact potential function.
2. $I(\mathcal{P}) = 0$ for every finite closed path $\mathcal{P}$.
3. $I(\mathcal{P}) = 0$ for every finite simple closed path $\mathcal{P}$.
4. $I(\mathcal{P}) = 0$ for every finite simple closed path $\mathcal{P}$ of length 4.

"The following are equivalent" means $1 \iff 2 \iff 3 \iff 4$.



Monderer and Shapley: show $2 \implies 3 \implies 4$, show $1 \iff 2$ and show $4 \implies 2$.

hard

Exercise 3 - Verifying a game is potential

Consider prisoner's dilemma. Verify it is a potential game using the characterization on the previous slide.

$\gamma(4) =$

$\gamma(3)$ $\qquad$ $\gamma(0)$

$$\begin{array}{c|cc} & \text{confess} & \text{silent} \\ \hline \text{confess} & (5,5) & (0,10) \\ \text{silent} & (10,0) & (1,1) \end{array}$$

$\gamma(2)$ $\qquad$ $\gamma(1)$

let $\gamma(0) = (\text{confess}, \text{silent})$

$$I(P) = J_1(\gamma(1)) - J_1(\gamma(0))$$
$$+ J_2(\gamma(2)) - J_2(\gamma(1)) +$$
$$J_1(\gamma(3)) - J_1(\gamma(2)) +$$
$$J_2(\gamma(4)) - J_2(\gamma(3))$$

Exercise 4 - convergence of better-response (better-reply) dynamics

A path is an **improvement** path if

$$\gamma(k) = (\gamma_{i_k}, \gamma_{-i_k}) \text{ with } \gamma_{i_k} \neq \gamma_{i_{k-1}} \iff J_{i_k}(\gamma(k)) < J_{i_k}(\gamma(k-1))$$

where i_k is the player that changes its strategy at step k.

Proposition

Consider a finite action game. Every improvement path of is finite. If the game is potential, the improvement path terminates at a Nash equilibrium.

See, for example, [Hespanha, Proposition 13.1].

An important class of potential games: congestion games

- Consider a game with a set $\mathcal{N} = \{1, 2, \ldots, N\}$ of players, and $\mathcal{M} = \{1, \ldots, M\}$ resources (road segments, communication routes, ...)
- Each strategy corresponds to a subset of resources that the player will use:

$$\text{for example} \qquad \gamma_i^{(1)} = \{4\}, \;\; \gamma_i^{(2)} = \{2, 4, 6\}, \;\; \ldots$$

- Denote the load on resource r as the number of players who use it

$$\ell_r(\gamma) := |\{i \in \mathcal{N} \mid r \in \gamma_i\}|$$

- The cost for each player depends on the load on the resources she is using

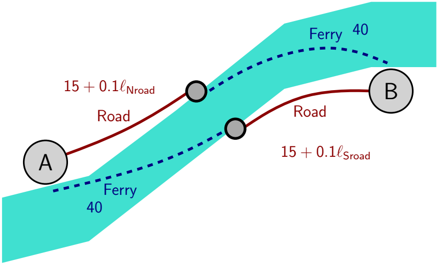
$$J_i(\gamma) = \sum_{r \in \gamma_i} f_r(\ell_r(\gamma))$$

1. The function f_r is resource-specific and non-decreasing
2. Each player who chooses a given resource experience the same cost corresponding to the resource as other players choosing the same resource

What are the domain and range of ℓ_r and f_r?

Examples: transportation or communication networks ...

Example: Traffic routing



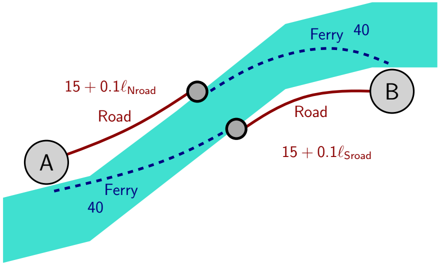
There are two ways to reach city B from city A, and both include some driving, and a trip on the ferry.

The two paths are perfectly equivalent, the only difference is whether you first drive, or take the ferry.

The time needed for the trip **depends on what other travellers do**.

- The **ferry** time is constant, 40 minutes
- The **road** time depends on the number of cars on the road.

Traffic routing as a congestion game



Formulation as a congestion game.

Each traveller is a Player.

Each path is a resource.

$f_r(\ell_r)$ described the time spent on r

- Each Player can decide to take the **North** or the **South** path.

$$\gamma_i = \begin{cases} \{\text{N road}, \text{N ferry}\} & \text{North} \\ \{\text{S road}, \text{S ferry}\} & \text{South} \end{cases}$$

- All players have **identical cost function**

$$J_i(\gamma_i, \gamma_{-i}) = \begin{cases} 40 + 15 + 0.1\ell_{\text{N road}} & \text{if } \gamma_i = \text{N} \\ 40 + 15 + 0.1\ell_{\text{S road}} & \text{if } \gamma_i = \text{S} \end{cases}$$

Every congestion game is an exact potential game

Theorem

The following is an exact potential function for congestion games.

$$P(\gamma) = \sum_{r=1}^{M} \sum_{k=1}^{\ell_r(\gamma)} f_r(k).$$

Proof

Consider a player i, and two joint pure strategies $\gamma = (\gamma_i, \gamma_{-i})$ and $\gamma' = (\gamma_i', \gamma_{-i})$. It suffices to show that

$$P(\gamma') - P(\gamma) = J_i(\gamma') - J_i(\gamma).$$

Note that

- $\ell_p(\gamma') = \ell_p(\gamma) - 1$ for every resource $p \in \gamma_i \backslash \gamma_i'$
- $\ell_q(\gamma') = \ell_q(\gamma) + 1$ for every resource $q \in \gamma_i' \backslash \gamma_i$
- $\ell_r(\gamma') = \ell_r(\gamma)$ for every other resource r. $r \in \gamma_i$ & $r \in \gamma_i'$

$\gamma = (\gamma^{i}, \gamma^{-i})$

Proof (cont.) Recall $\gamma' = (\gamma'^{i}, \gamma^{-i})$

$$
\begin{aligned}
P(\gamma') - P(\gamma) &= \sum_{r=1}^{M} \sum_{k=1}^{\ell_r(\gamma')} f_r(k) - \sum_{r=1}^{M} \sum_{k=1}^{\ell_r(\gamma)} f_r(k) \\
&= \sum_{q \in \gamma_i' \setminus \gamma_i} f_q(\ell_q(\gamma')) - \sum_{p \in \gamma_i \setminus \gamma_i'} f_p(\ell_q(\gamma))
\end{aligned}
$$

which corresponds exactly to the difference between $J_i(\gamma')$ and $J_i(\gamma)$, namely,

$$
\begin{aligned}
J_i(\gamma') - J_i(\gamma) &= \sum_{l \in \gamma_i'} f_l(\ell_l(\gamma')) - \sum_{l \in \gamma_i} f_l(\ell_l(\gamma)) \\
&= \sum_{q \in \gamma_i' \setminus \gamma_i} f_q(\ell_q(\gamma')) - \sum_{p \in \gamma_i \setminus \gamma_i'} f_p(\ell_q(\gamma))
\end{aligned}
$$

Consequently, congestion games admit a pure Nash equilibrium.

Understanding the potential function of the congestion games

MK: ask students to derive the social welfare function. then they see they are not aligned. for the left-hand be: 2c(1) + 3c(3); for the right-hand-side will be 6c(2)s

I'll come back to this in a moment

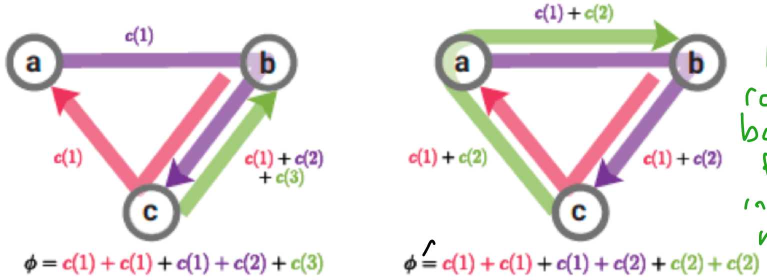
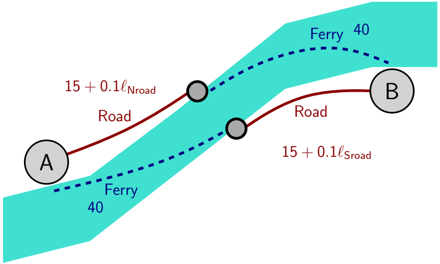


FIGURE 4.10. In this example, the cost is the same on all three roads, and is $c(n)$ if the number of drivers is n. The figure shows the potential ϕ before and after the player going from c to b switches from the direct path to the indirect path through a. The change in potential is the change in cost experienced by this player.

$\phi' - \phi = 2c(2) - c(3)$,

Figure: From [Karlin and Peres, *Game Theory, Alive*]

Traffic routing: properties of the pure strategy Nash equilibrium



What are the pure Nash equilibria?

We consider a **population** of $N = 200$ travelers.

- Suppose the first 100 players choose the North path and the second 100 players choose the South path.
- Travel cost of each player:

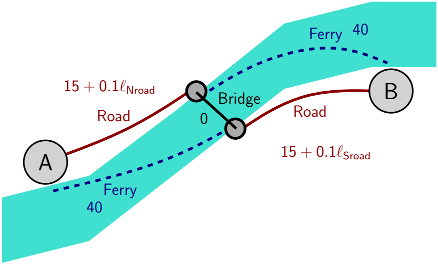
$$J_i(\gamma_i, \gamma_{-i}) = 40 + 15 + 0.1 \cdot \frac{200}{2} = 65 \text{ minutes}$$

- Can any player improve the outcome by **unilaterally** deviating from the ~~NE~~?

No, because, road cost will be above? solution

$101 \times .1$

Braess paradox through an example



Assume a bridge is build, to help reduce traffic.

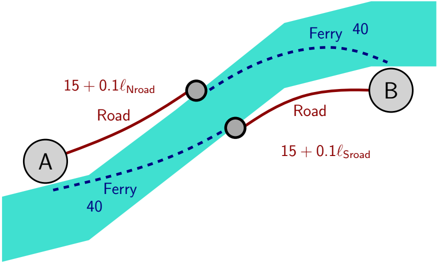
It takes no time to cross the bridge, allowing to go from city A to city B without taking the ferry.

all travelers avoid the ferry.

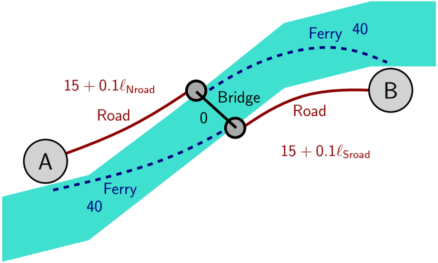
$$J_i(\gamma^*) = 2\,(15 + 0.1 \cdot 200) = 70 \text{ minutes}$$

Can you improve your outcome by **unilaterally** deviate from the above?
No, road + ferry now takes $40 + 15 + 0.1 \cdot 200 = 75$ minutes!

Braess paradox - where is the paradox?



$J_i^{\text{NE}} = 65$ minutes



$J_i^{\text{NE}} = 70$ minutes

With the new link in the **transportation graph**

- the original choice (road + ferry) is still present
- the new link is intensively used
- all agents experience higher cost!

The New York Times

What if They Closed 42d Street and Nobody Noticed?

25 December 1990

“On Earth Day this year, New York City’s Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. [...] But to everyone’s surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.”

And many other real-life cases in road traffic, data networks, etc.

TECHNOLOGY

The Perfect Selfishness of Mapping Apps

Apps like Waze, Google Maps, and Apple Maps may make traffic conditions worse in some areas, new research suggests.

Why Some Cities Have Had Enough of Waze

We are not just facing data security and privacy issues caused by the mass digitalization of our ordinary lives, we are also facing physical danger and lifestyle degradation in areas we would never had imagined that technology would affect negatively. Photo: HT/File

Of Map Apps and Traffic Jams

For the Good of Society — and Traffic! — Delete Your Map App

- https://nymag.com/intelligencer/2017/12/waze-and-google-maps-create-traffic-in-cities.html
- https://www.usnews.com/news/national-news/articles/2018-05-07/why-some-cities-have-had-enough-of-waze
- https://www.livemint.com/Opinion/dcZr24gKTno410a6xh0RIN/Of-Map-Apps-and-Traffic-Jams.html
- https://www.theatlantic.com/technology/archive/2018/03/mapping-apps-and-the-price-of-anarchy/555551/

Distributed welfare games

The above observations motivate considering how to optimize total travel time...

Welfare function

In a N-person game, let $\gamma_i \in \Gamma_i$ be the strategy played by agent i.

Let $\gamma \in \Gamma := \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_N$ be the system-wide strategy.

A **welfare cost** $W : \Gamma \to \mathbb{R}$ is a measure of efficiency of each strategy for the social cost of the population of agents.

If the **individual cost** that player i wants to minimize is $J_i(\gamma)$, the welfare function can be for example,

$$W(\gamma) = \sum_i J_i(\gamma) \qquad W(\gamma) = \max_i J_i(\gamma) \qquad W(\gamma) = \sum_i \log J_i(\gamma)$$

Price of Anarchy

The **Price of Anarchy** is defined as the ratio

$$\text{PoA} := \frac{\max_{\gamma\in\Gamma_{\text{NE}}} W(\gamma)}{\min_{\gamma\in\Gamma} W(\gamma)}$$

where Γ is the set of all possible strategies for all agents, while Γ_{NE} is the set of all strategies which are NE.

sum of travel times of all player

In **Braess** paradox example, assume $W(\gamma) = \sum_{i=1}^{N} J_i(\gamma)$.

64.375

$$PoA = \frac{70}{64.xxxx} = 1\cancel{09\%}$$

1.08

Exercise: How would you formulate the problem of optimizing the social welfare function?

75 people take Ferry + south road

" " " Ferry + north road

50 " " bridge + north + south road

Historical notes

- Robert W. Rosenthal: defined congestion games in 1973 as a class of games that have a pure strategy equilibrium.
- Monderer and Shapley: defined potential games and showed congestion games are equivalent up to an *isomorphism* to potential games (1996). See their paper for the definition of game isomorphism.
- Shapley won the Nobel prize in economics for his contributions to game theory
  - potential games, matching algorithms, Markov games, cooperative games
- Shapley and Nash were students of the same doctoral thesis (A. W. Tucker)

Lloyd Shapley

Bounding the price of anarchy and mechanism design are active topics in engineering and computer science, see for example here and here, respectively.

Summary

- Best-response dynamics and its convergence properties
- Potential games: ordinal and exact potential functions
- Maximizers of potential function and pure Nash equilibrium
- Tools to determine if a game is potential
- Paths, Improvement paths in playing the game
- Convergence of better-reply dynamics in finite action games
- Congestion games as an important class of potential games
- Social welfare optimization
- Braess Paradox
- Price of Anarchy
- Game design for reducing price of anarchy